$$Y_{k2} - \sum_{n=1}^{\infty} \{Y_{n1} \operatorname{Im} [\varphi_n (\bar{b}_k) - \varphi_n (b_k)] + Y_{n2} \operatorname{Re} [\varphi_n (b_k) - \varphi_n (\bar{b}_k)]\} = - \operatorname{Im} \psi_k$$

where

$$\varphi_n (b_k) = b_k^{-1} \exp [l (b_k + b_n)] T (- b_n, b_k), \quad \psi_k = 2b_k^{-1} [t (b_k) - c_k]$$

The second condition in (4.3) is satisfied automatically. In the general case $A_k, B_k \sim O(e^{-\pi kl})$ if $f_1(r) = f_2(r) = 0$, which corresponds to compression of an infinite cylinder by two semi-infinite collars $A_k, B_k \sim O(ke^{-2\pi kl})$.

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BUCKLING OF PLATES MADE OF A NEO-HOOKEAN MATERIAL IN THE CASE OF AFFINE INITIAL DEFORMATION

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L. M. ZUBOV
(Leningrad)

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We construct two-dimensional equations describing the bending bifurcation of equilibrium of plates made of a neo-Hookean material, for the case of homogeneous initial deformation. We derive three-dimensional equations of neutral equilibrium for this material. A variational principle which is equivalent to the differential equations of neutral equilibrium and analogous to the the Reissner's principle in the classical theory of elasticity, is established. We use this variational principle to derive two-dimensional equations of buckling of plates by approximating the variations in the values of the unknown functions in the normal direction. The cases of buckling of a uniformly compressed circular plate and of a rectangular plate under a combined load are used as examples. An exact solution of three-dimensional equations of neutral equilibrium is obtained for a circular cylinder compressed over its lateral surface, with axial symmetry present, and compared with the corresponding two-dimensional result.

1. Equations of neutral equilibrium for a neo-Hookean material. Specific potential energy of deformation is given for a neo-Hookean material by the following expression:

$$W = c_1 (I_1 - 3), \qquad c_1 = \text{const}$$

The material is incompressible, i.e. when it is deformed, the condition

$$I_3(G^\times) = 1 \tag{1.1}$$

where I_1 and I_3 are the first and third invariant of the tensor $G^\times$, must be fulfilled. Here $G^\times$ is the Cauchy's measure of deformation [1 and 2]. Finger's form [1] of the equation of state of an isotropic elastic body yields the following expression for the stress tensor for a neo-Hookean material:

$$T = 2c_1 \nabla \mathbf{R}^T \cdot \nabla \mathbf{R} + 2c_{-1} E \tag{1.2}$$

Here $\mathbf{R}$ is the radius vector of a point of the deformed body, ∇ is the nabla operator in the metric of the undeformed state and E is a unit tensor. The quantity c_{-1} in (1.2) is an undefined function of the strain invariants and in the case of a specific problem it should be obtained from the equations of equilibrium and the condition of incompressibility. Further from the formula [2]

$$D = \sqrt{I_3}\,(\nabla \mathbf{R}^T)^{-1} \cdot T$$

where D is the Piola stress tensor, we obtain the following expression for a neo-Hookean material

$$D = 2c_1 \nabla \mathbf{R} + 2c_{-1} (\nabla \mathbf{R}^T)^{-1} \tag{1.3}$$

The neutral equilibrium equations for an elastic body with external dead loading can be written in the metric of the undeformed state as follows [2]:

$$\nabla \cdot D^\bullet = 0 \text{ in the volume } V; \quad \mathbf{n} \cdot D^\bullet = 0 \text{ on } o_1, \quad \mathbf{w} = 0 \text{ on } o_2 \tag{1.4}$$

$$D^\bullet = \left\{ \frac{d}{d\eta} D(\mathbf{R}^\circ + \eta \mathbf{w}) \right\}_{\eta=0}$$

Here $\mathbf{R}^\circ$ is the radius vector of a body point corresponding to the initial state of equilibrium which is under investigation for stability, o_1 denotes the part of the surface of the undeformed body at which the external forces are applied and o_2 denotes the part of the surface on which the displacements are given. For a neo-Hookean material (1.3) yields

$$\tfrac{1}{2} c_1^{-1} D^\bullet = \nabla \mathbf{w} + p(\nabla R^{\circ T})^{-1} + m(\nabla \mathbf{R}^{\circ T})^{-1} \cdot \nabla \mathbf{w}^T \cdot (\nabla \mathbf{R}^{\circ T})^{-1} \tag{1.5}$$

$$p = (c_{-1})^\bullet / c_1, \qquad m = -c_{-1}{}^\circ / c_1$$

We note that both, p and the vector of additional displacement w, are unknown functions of the coordinates. The quantity p can be found from the equations of equilibrium (1.4) and the condition of incompressibility

$$I_3{}^\bullet = 0, \qquad \text{or} \qquad (\nabla \mathbf{R}^\circ)^{-1} \cdot\cdot \nabla \mathbf{w} = 0 \tag{1.6}$$

In the case of affine initial deformation we introduce, for convenience, the rotated displacement vector $\mathbf{w}' = \mathbf{w} \cdot A^{\circ T}$ and the tensor $D^{\bullet\prime} = D^\bullet \cdot A^{\circ T}$, where $A^\circ = G^{\times\circ -1/2} \cdot \nabla \mathbf{R}^\circ$ is the rotation tensor of the principal axes of the initial deformation. Then in place of (1.5) and (1.6) we obtain

$$\tfrac{1}{2} c_1^{-1} D^{\bullet\prime} = \nabla \mathbf{w}' + p G^{\times\circ -1/2} + m G^{\times\circ -1/2} \cdot \nabla \mathbf{w}'^T \cdot G^{\times\circ -1/2} \tag{1.7}$$

$$\nabla \cdot G^{\times\circ -1/2} \cdot \mathbf{w}' = 0 \tag{1.8}$$

Since the tensor A° is constant when the initial deformation is affine, Eqs. (1.4) are obviously equivalent to

$$\nabla \cdot D^{\bullet\prime} = 0 \text{ in } V, \quad \mathbf{n} \cdot D^{\bullet\prime} = 0 \text{ on } o_1, \quad \mathbf{w}' = 0 \text{ on } o_2$$

Inserting (1.7) we obtain the following differential equations of neutral equilibrium in terms of displacements for a neo-Hookean material in the case of affine initial deformation

$$\nabla^2 \mathbf{w}' + \mathrm{G}^{\times\circ-1/2} \cdot \nabla p = 0 \qquad (1.9)$$

In deriving this we have used the fact that the condition of incompressibility (1.8) implies

$$\nabla \cdot \mathrm{G}^{\times\circ-1/2} \cdot \nabla \mathbf{w}'^T \cdot \mathrm{G}^{\times\circ-1/2} = \nabla \cdot \mathrm{G}^{\times\circ-1/2} (\nabla \cdot \mathrm{G}^{\times\circ-1/2} \cdot \mathbf{w}') = 0$$

Only the affine initial deformation is considered in what follows. In order to simplify the notation we shall omit the prime as well as dispense with the vector $\mathbf{w}'$ and the tensor $\mathrm{D}^{\circ\prime}$. We shall also denote the vector $\mathbf{w}^\circ \mathrm{A}^{\circ T}$ and the tensor $\mathrm{D}^\bullet \cdot \mathrm{A}^{\circ T}$ by $\mathbf{w}$ and D°, respectively. Finally, we shall omit the index $^\circ$ where it refers to the initial state of stress-strain.

2. Variational formulation of the problem on bifurcation of equilibrium for a neo-Hookean material in the case of affine initial deformation. Let us consider the following functional over the vector $\mathbf{w}$ and tensor $\mathrm{D}^\bullet$, both of which we regard as independent functions of the coordinates:

$$\Phi = \iiint_v \Big\{ \partial_{sk}^\bullet \partial_s u_k - \frac{1}{4c_1} \Big[\frac{\sqrt{G_s G_k}}{G_s G_k - m^2} \Big(\frac{G_s G_k}{2} \partial_{sk}^{\bullet 2} + \frac{\sqrt{G_s G_k}}{2} \partial_{ks}^{\bullet 2} -$$

$$- m \partial_{sk}^\bullet \partial_{ks}^\bullet \Big) - \Big(\sum_{n=1}^{3} \frac{1}{G_n + m} \Big)^{-1} \Big(\sum_{n=1}^{3} \frac{\sqrt{G_n} \partial_{nn}^\bullet}{G_n + m} \Big)^2 \Big] \Big\} d\tau \qquad (2.1)$$

$$\mathrm{D}^\bullet = \partial_{sk}^\bullet \mathbf{e}_s \mathbf{e}_k, \qquad \mathbf{w} = u_k \mathbf{e}_k, \qquad \partial_s^\bullet = \frac{\partial}{\partial x_s}$$

Here the integration is performed over the volume of the body in its undeformed state, G_s denotes the principal values of the tensor $\mathrm{G}^\times$ in the initial state of stress, $\partial_{sk}^\bullet$ denote the components of the tensor $\mathrm{D}^\bullet$ with respect to the basis of the principal directions $\mathbf{e}_k$ of the tensor $\mathrm{G}^\times$ in the initial stressed state, u_k are the components of the vector $\mathbf{w}$ with respect to the basis $\mathbf{e}_k$, and x_s are Cartesian coordinates of the undeformed body, their axes coinciding with the unit vectors $\mathbf{e}_s$. The summation in (2.1) is understood to be performed over the indices $s,\ k = 1,\ 2,\ 3$.

We shall show that the functions $\mathbf{w}$ twice continuously differentiable with respect to coordinates and satisfying the condition $\mathbf{w} = 0$ on o_2 and all continuously differentiable tensors $\mathrm{D}^\bullet$ impart a stationary value to the functional Φ if and only if they satisfy

$$\nabla \cdot \mathrm{D}^\bullet = 0, \quad \nabla \cdot \mathrm{G}^{\times-1/2} \cdot \mathbf{w} = 0, \quad \frac{1}{2c_1} \mathrm{D}^\bullet = \nabla \mathbf{w} + m \mathrm{G}^{\times-1/2} \cdot \nabla \mathbf{w}^T \cdot \mathrm{G}^{\times-1/2} + p \mathrm{G}^{\times-1/2} \qquad (2.2)$$

where p is a continuously differentiable function of the coordinates, and the following boundary conditions on o_1:

$$\mathbf{n} \cdot \mathrm{D}^\bullet = 0$$

Let us compute the variation of the functional Φ

$$\delta\Phi = \iiint_v \Big\{ \partial_{sk}^\bullet \partial_s \delta u_k + {}^1\!/_2 \partial_s u_k \delta\partial_{sk}^\bullet + {}^1\!/_2 \partial_k u_s \delta\partial_{ks}^\bullet -$$

$$- \frac{1}{4c_1} \Big[\frac{\sqrt{G_s G_k}}{G_s G_k - m^2} (\sqrt{G_s G_k}\, \partial_{sk}^\bullet \delta\partial_{sk}^\bullet + \sqrt{G_s G_k}\, \partial_{ks}^\bullet \delta\partial_{ks}^\bullet -$$

$$- m \partial_{sk}^\bullet \delta\partial_{ks}^\bullet - m \partial_{ks}^\bullet \delta\partial_{sk}^\bullet) -$$

$$-2\left(\sum_{n=1}^{3}\frac{1}{G_n+m}\right)^{-1}\sum_{n=1}^{3}\frac{\sqrt{G_n}\partial_{nn}^{\bullet}}{G_n+\bar{m}}\sum_{n=1}^{3}\frac{\sqrt{G_n}\,\delta\partial_{nn}^{\bullet}}{G_n+\bar{m}}\Big]\Big\}d\tau$$

Since by definitions the variations δu_k are arbitrary, we arrive after integrating by parts, at

$$\partial_s\partial_{sk}^{\bullet}=0 \text{ in } v^{\bullet}, \qquad n_s\partial_{sk}^{\bullet}=0 \text{ on } o_1 \qquad (k=1,2,3) \tag{2.3}$$

Further, equating to zero the coefficients accompanying the variations $\delta\partial_{sk}$ and $\delta\partial_{ks}$, we obtain

$$\frac{1}{2}\partial_s u_k-\frac{1}{4c_1}\left[\frac{\sqrt{G_sG_k}}{G_sG_k-m^2}(\sqrt{G_sG_k}\,\partial_{sk}^{\bullet}-m\partial_{ks}^{\bullet})\right]=0$$

$$\frac{1}{2}\partial_k u_s-\frac{1}{4c_1}\left[\frac{\sqrt{G_sG_k}}{G_sG_k-m^2}(\sqrt{G_sG_k}\,\partial_{ks}^{\bullet}-m\partial_{sk}^{\bullet})\right]=0 \tag{2.4}$$

($s\neq k$, not to be summed over s, k!)

$$\partial_s u_s=\frac{1}{4c_1}\left[\frac{G_s}{G_s+m}\partial_{ss}^{\bullet}-2\left(\sum_{n=1}^{3}\frac{1}{G_n+m}\right)^{-1}\sum_{n=1}^{3}\frac{\sqrt{G_n}\partial_{nn}^{\bullet}}{G_n+m}\,\frac{\sqrt{G_s}}{G_s+m}\right. \tag{2.5}$$

(not to be summed over s!)

Solving (2.4) for $\partial_{sk}^{\bullet}$ we obtain

$$\frac{1}{2c_1}\partial_{sk}^{\bullet}=\partial_s u_k+\frac{m}{\sqrt{G_sG_k}}\partial_k u_s \qquad (s\neq k, \text{ not to be summed over } s, k) \tag{2.6}$$

Rewriting (2.5) we obtain the following expressions for the diagonal components of the tensor $D^{\bullet}$:

$$\frac{\sqrt{G_s}}{G_s+m}\partial_{ss}^{\bullet}-\frac{1}{G_s+m}\left(\sum_{n=1}^{3}\frac{1}{G_n+m}\right)^{-1}\sum_{k=1}^{3}\frac{\sqrt{G_k}\partial_{kk}^{\bullet}}{G_k+m}=2c_1\frac{\partial_s u_s}{\sqrt{G_s}} \tag{2.7}$$

(not to be summed over $s=1,2,3$)

Adding the three equations of (2.7) we find that the sum of the left parts is identically zero, and this yields the condition of incompressibility

$$\sum_{s=1}^{3}\frac{\partial_s u_s}{\sqrt{G_s}}=0 \tag{2.8}$$

When the above condition holds, a solution of (2.7) exists but is not unique and has the form

$$\frac{1}{2c_1}\partial_{ss}^{\bullet}=\frac{G_s+m}{G_s}\partial_s u_s+\frac{p}{\sqrt{G_s}} \qquad (\text{not to be summed over } s!) \tag{2.9}$$

where p is an undefined function of the coordinates. Combining (2.6) and (2.9) we obtain

$$\frac{1}{2c_1}\partial_{sk}^{\bullet}=\partial_s u_k+\frac{m}{\sqrt{G_sG_k}}\partial_k u_s+p\delta_{sk}\frac{1}{\sqrt{G_s}} \qquad (\text{not to be summed over } s, k=1,2,3!) \tag{2.10}$$

Since Eqs. (2.3), (2.8) and (2.10) agree with (2.2), our assertion is proved.

3. Derivation of two-dimensional equations of plate buckling. Assuming that the initial deformation in the plate represents a plane affine transformation accompanied by uniform elongation in the z-direction we use Eqs. (1.8) and (1.9) of neutral equilibrium to conclude that there are two mutually independent types of forms of bifurcation: symmetric with respect to the neutral plane $z=0$ and antisymmetric, i.e. flexural. For the flexural forms of buckling we shall use, as in [3], the following approximate expressions for the displacement vector $\mathbf{w}$ and the tensor $\mathbf{D}^{\cdot}$ in terms of the z-coordinate

$$\mathbf{w} = \mathbf{w}_1 z + w_0 \mathbf{i}_3, \qquad \mathbf{w}_1 = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2$$
$$\partial_{sk}^{\cdot} = \frac{12}{h^3} M_{sk} z \quad (s,\ k = 1,\ 2), \qquad \partial_{33}^{\cdot} = 0 \tag{3.1}$$
$$\partial_{13}^{\cdot} = \frac{3}{2h}\left[1 - \left(\frac{2z}{h}\right)^2\right] V_{13} + \frac{1}{h} V_{13}', \qquad \partial_{31}^{\cdot} = \frac{3}{2h}\left[1 - \left(\frac{2z}{h}\right)^2\right] V_{31}$$
$$\partial_{23}^{\cdot} = \frac{3}{2h}\left[1 - \left(\frac{2z}{h}\right)^2\right] V_{23} + \frac{1}{h} V_{23}', \qquad \partial_{32}^{\cdot} = \frac{3}{2h}\left[1 - \left(\frac{2z}{h}\right)^2\right] V_{32}$$

Introducing two-dimensional vectors

$$V_{13}\mathbf{e}_1 + V_{23}\mathbf{e}_2 = \mathbf{V}_1, \quad V_{13}'\mathbf{e}_1 + V_{23}'\mathbf{e}_2 = \mathbf{V}_3, \quad V_{31}\mathbf{e}_1 + V_{32}\mathbf{e}_2 = \mathbf{V}_2$$

we obtain

$$\mathbf{M} = M_{sk}\mathbf{e}_s\mathbf{e}_k = \int_{-h/2}^{h/2} \partial_{sk}^{\cdot}\mathbf{e}_s\mathbf{e}_k z\, dz$$
$$\mathbf{V}_2 = \int_{-h/2}^{h/2} \partial_{3s}^{\cdot}\mathbf{e}_s\, dz, \qquad \mathbf{V}_1 + \mathbf{V}_3 = \int_{-h/2}^{h/2} \partial_{s3}^{\cdot}\mathbf{e}_s\, dz \qquad (s,\ k = 1,\ 2)$$

Here h denotes the thickness of the plate in its undeformed state. Using the variational principle formulated in Sect. 2, we obtain the functions $w_0, \mathbf{w}_1, \mathbf{M}, \mathbf{V}_1, \mathbf{V}_2$ and $\mathbf{V}_3$ of two variables $(x,\ y)$ from the condition of stationarity of the functional (2.1). Let us substitute (3.1) into (2.1), integrate the result with respect to z and calculate the variation

$$\delta\Phi = \iint_S \Big\{-(\nabla\cdot\mathbf{M})\cdot\delta\mathbf{w}_1 - \delta w_0\partial_1(V_{13} + V_{13}') - \delta w_0\partial_2(V_{23} + V_{23}') + \delta u_1 V_{31} +$$
$$+ \delta u_2 V_{32} + \delta M_{sk}\left(\frac{1}{2}\partial_s u_k - \frac{12}{4c_1h^3}\frac{G_sG_k}{G_sG_k - m^2}M_{sk} + \frac{12m}{4c_1h^3}\frac{\sqrt{G_sG_k}}{G_sG_k - m^2}M_{ks}\right) +$$
$$+ \delta M_{ks}\left(\frac{1}{2}\partial_k u_s - \frac{12}{4c_1h^3}\frac{G_sG_k}{G_sG_k - m^2}M_{ks} + \frac{12m}{4c_1h^3}\frac{\sqrt{G_sG_k}}{G_sG_k - m^2}M_{sk}\right) +$$
$$+ \sum_{s=1}^{2}\delta M_{ss}\left[\frac{24}{4c_1h^3}\frac{\sqrt{G_s}}{G_s + m}\left(\sum_{k=1}^{3}\frac{1}{G_k + m}\right)^{-1}\sum_{n=1}^{2}\frac{\sqrt{G_n}M_{nn}}{G_n + m}\right] +$$
$$+ \delta V_{13}\left[\partial_1 w_0 - \frac{\sqrt{G_1G_3}}{4c_1h(G_1G_3 - m^2)}\left(\frac{12}{5}\sqrt{G_1G_3}V_{13} + 2\sqrt{G_1G_3}V_{13}' - \frac{12}{5}mV_{31}\right)\right] +$$
$$+ \delta V_{23}\left[\partial_2 w_0 - \frac{\sqrt{G_2G_3}}{4c_1h(G_2G_3 - m^2)}\left(\frac{12}{5}\sqrt{G_2G_3}V_{23} + 2\sqrt{G_2G_3}V_{23}' - \frac{12}{5}mV_{32}\right)\right] +$$
$$+ \delta V_{13}'\left[\partial_1 w_0 - \frac{\sqrt{G_1G_3}}{4c_1h(G_1G_3 - m^2)}\left(2\sqrt{G_1G_3}V_{13} + 2\sqrt{G_1G_3}V_{13}' - 2mV_{31}\right)\right] +$$
$$+ \delta V_{23}'\left[\partial_2 w_0 - \frac{\sqrt{G_2G_3}}{4c_1h(G_2G_3 - m^2)}\left(2\sqrt{G_2G_3}V_{23}' + 2\sqrt{G_2G_3}V_{23}' - 2mV_{32}\right)\right] +$$
$$+ \delta V_{31}\left[u_1 - \frac{\sqrt{G_1G_3}}{4c_1h(G_1G_3 - m^2)}\left(\frac{12}{5}\sqrt{G_1G_3}V_{31} - \frac{12}{5}mV_{13} - 2mV_{13}'\right)\right] +$$
$$+ \delta V_{32}\left[u_2 - \frac{\sqrt{G_2G_3}}{4c_1h(G_2G_3 - m^2)}\left(\frac{12}{5}\sqrt{G_2G_3}V_{32} - \frac{12}{5}mV_{23} - 2mV_{23}'\right)\right]\Big\}\, do +$$

$$+ \oint_{\gamma} [\mathbf{n} \cdot \mathrm{M} \cdot \delta \mathbf{w}_1 + \mathbf{n} \cdot (\mathbf{V}_1 + \mathbf{V}_3)\, \delta w_0]\, ds = 0$$

Here S is the neutral plane of the plate, γ is its boundary, n is the normal to this boundary and ∇ is the two-dimensional nabla operator.

Arbitrariness of the variations δM_{sk} and δM_{ks} implies that

$$\sqrt{G_s G_k}\, M_{sk} - m M_{ks} = \frac{c_1 h^3}{6} \frac{G_s G_k - m^2}{\sqrt{G_s G_k}} \partial_s u_k \qquad (3.2)$$

$$- m M_{sk} + \sqrt{G_s G_k}\, M_{ks} = \frac{c_1 h^3}{6} \frac{G_s G_k - m^2}{\sqrt{G_s G_k}} \partial_k u_s$$

($s \neq k$ not to be summed over $s, = 1, 2!$)

$$\frac{\sqrt{G_s}}{G_s + m} M_{ss} - \frac{1}{G_s + m} \left(\sum_{k=1}^{3} \frac{1}{G_k + m} \right)^{-1} \sum_{n=1}^{2} \frac{\sqrt{G_n}\, M_{nn}}{G_n + m} = \frac{c_1 h^3}{6} \frac{\partial_s u_s}{\sqrt{G_s}} \qquad (3.3)$$

(not to be summed over $s, k = 1, 2!$)

From (3.2) we find

$$M_{sk} = \frac{c_1 h^3}{6} \left(\partial_s u_k + \frac{m}{\sqrt{G_s G_k}} \partial_k u_s \right) \qquad (s \neq k) \qquad (3.4)$$

The system (3.3) for the diagonal components of the tensor M admits a unique solution when the right hand sides are arbitrary, namely

$$M_{ss} = \frac{c_1 h^3}{6} \left[\left(1 + \frac{m}{G_s} \right) \partial_s u_s + \frac{G_3 + m}{\sqrt{G_s}} \sum_{k=1}^{2} \frac{\partial_k u_k}{\sqrt{G_k}} \right] \qquad (3.5)$$

(not to be summed over $s = 1, 2!$)

It follows therefore that in contrast to the general case of Sect. 2, the kinematic quantities introduced here must be regarded as independent. Combining (3.4) and (3.5) we obtain

$$M_{sk} = \frac{c_1 h^3}{6} \left[\partial_s u_k + \frac{m}{\sqrt{G_s G_k}} \partial_k u_s + \delta_{sk} \frac{G_3 + m}{\sqrt{G_s}} \sum_{k=1}^{2} \frac{\partial_k u_k}{\sqrt{G_k}} \right] \qquad (3.6)$$

(not to be summer over $s, k = 1, 2$)

Let us consider such an initial deformation, which leaves the plate faces $z = \pm h / 2$ load-free. Then from (1.3) we obtain

$$m = - c_{-1} / c_1 = G_3$$

Moreover, the initial deformation satisfies the condition of incompressibility

$$G_1 G_2 G_3 = 1$$

Taking this into account we can write (3.6) in the invariant form as

$$\mathrm{M} \cdot \mathrm{G}^{\times^{1/2}} = \frac{c_1 h^3}{6} [\nabla \mathrm{G}^{\times} \cdot \mathbf{w}_2 + G_3 \nabla \mathbf{w}_2^{T} + 2 G_3 \nabla \cdot \mathbf{w}_2 \mathrm{E}_2] \qquad (3.7)$$

$$\mathrm{E}_2 = \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2, \qquad \mathbf{w}_2 = \mathrm{G}^{\times^{-1/2}} \cdot \mathbf{w}_1$$

Further, equating to zero the coefficients accompanying the variations of the vectors $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ and solving the resulting equations, we obtain

$$\mathbf{V}_2 \cdot \mathrm{G}^{\times^{1/2}} = {}^5/_3 c_1 h\, [\mathrm{G}^{\times} \cdot \mathbf{w}_2 + \sqrt{G_3} \nabla w_0] \qquad (3.8)$$

$$\mathbf{V}_1 + \mathbf{V}_3 = {}^1/_3 c_1 h\, [5 \sqrt{G_3} \mathbf{w}_2 + (6 \mathrm{E}_2 - G_3 \mathrm{G}^{\times^{-1}}) \cdot \nabla w_0] \qquad (3.9)$$

The arbitrariness of the variations of the kinematic quantities implies the following equations of equilibrium

$$\nabla \cdot \mathrm{M} \cdot \mathrm{G}^{\times^{1/2}} = \mathbf{V}_2 \cdot \mathrm{G}^{\times^{1/2}}, \qquad \nabla \cdot (\mathbf{V}_1 + \mathbf{V}_3) = 0 \tag{3.10}$$

Inserting the relations (3.7) – (3.9) into (3.10) we arrive at the following equations in terms of displacements, describing the bifurcation of equilibrium of the plate

$$\begin{gathered} {}^1/_{10}h^2 (\nabla^2 \mathrm{G}^\times \cdot \mathbf{w}_2 + 3G_3 \nabla\nabla \cdot \mathbf{w}_2) - \mathrm{G}^\times \cdot \mathbf{w}_2 - \sqrt{G_3} \nabla w_0 = 0 \\ 5\sqrt{G_3} \nabla \cdot \mathbf{w}_2 + \nabla \cdot (6\mathrm{E}_2 - G_3 \mathrm{G}^{\times-1}) \cdot \nabla w_0 = 0 \end{gathered} \tag{3.11}$$

A variety of boundary conditions follow clearly from the structure of the contour integral in the expression for $\delta\Phi$. We find that it is possible to obtain a separate fourth order differential equation for the flexure of the neutral surface w_0, from the system (3.11).

Let us multiply the first equation of (3.11) by $\sqrt{G_3}\mathrm{G}^{\times-1}$ and find the divergence of the left-hand side

$$\begin{gathered} {}^1/_{10}h^2 \sqrt{G_3} (\nabla^2 \nabla \cdot \mathbf{w}_2 + 3G_3 \nabla \cdot \mathrm{G}^{\times-1} \cdot \nabla\nabla \cdot \mathbf{w}_2) - \\ - \sqrt{G_3} \nabla \cdot \mathbf{w}_2 - G_3 \nabla \cdot \mathrm{G}^{\times-1} \cdot \nabla w_0 = 0 \end{gathered}$$

Inserting here $\nabla \cdot w_2$ from the second equation of (3.11) we obtain the required equation for w_0

$$\begin{gathered} {}^1/_{10}h^2 (\nabla^2 + 3G_3 \nabla \cdot \mathrm{G}^{\times-1} \cdot \nabla) \nabla \cdot (6E_2 - G_3 \mathrm{G}^{\times-1}) \cdot \nabla w_0 - \\ - 6\nabla \cdot (\mathrm{E}_2 - G_3 \mathrm{G}^{\times-1}) \cdot \nabla w_0 = 0 \end{gathered} \tag{3.12}$$

In addition we can easily see that the general solution for $\mathbf{w}_2$ can be written in the following form:

$$5\sqrt{G_3} \mathbf{w}_2 = \nabla \times \psi \mathbf{i}_3 + \boldsymbol{\varphi} - (6\mathrm{E}_2 - G_3 \mathrm{G}^{\times-1}) \cdot \nabla w_0$$

The scalar function ψ can be obtained from

$${}^1/_{10}h^2 \nabla^2 \psi - \psi = 0$$

Any particular solution of the equations

$${}^1/_{10}h^2 \nabla^2 \boldsymbol{\varphi} - \boldsymbol{\varphi} = {}^1/_{10}h^2 (\nabla^2 \mathrm{E}_2 + 3G_3 \mathrm{G}^{\times-1} \cdot \nabla\nabla) \cdot (6\mathrm{E}_2 - G_3 \mathrm{G}^{\times-1}) \cdot \nabla w_0 \tag{3.13}$$

$$- 6 (\mathrm{E}_2 - G_3 \mathrm{G}^{\times-1}) \cdot \nabla w_0, \qquad \nabla \cdot \boldsymbol{\varphi} = 0 \tag{3.14}$$

can be used as the vector $\boldsymbol{\varphi}$.

We note that the vector appearing in the right-hand side of (3.13) is solenoidal by virtue of Eq. (3.12). For example, when the plate is uniformly compressed

$$\mathrm{G}^\times = \mathrm{G}_1 \mathrm{E}_2 + \mathrm{G}_3 \mathbf{i}_3 \mathbf{i}_3$$

the solution of (3.13) and (3.14) is

$$\boldsymbol{\varphi} = - {}^1/_{10}h^2 (1 + 3G_1^{-3}) (6 - G_1^{-3}) \nabla\nabla^2 w_0 + 6 (1 - G_1^{-3}) \nabla w_0$$

Assuming now that the initial deformation in the plate is very small, we linearize the differential equation (3.12) with respect to this deformation. When the deformations are small, we have

$$\mathrm{G}^{\times-1} = \mathrm{E} - 2\varepsilon$$

Here ε is the linear deformation tensor. By virtue of the condition of incompressibility the linear approximation

$$G_3 = 1 - 2\mathrm{E}_2 \cdot\cdot \varepsilon$$

This inserted into (3.12) yields

$$\nabla^4 w_0 - {}^1/_2 h^{-2} (12 + {}^{11}/_5 h^2 \nabla^2) \nabla \cdot (\varepsilon + \mathrm{E}_2 \cdot\cdot \varepsilon \mathrm{E}_2) \nabla \cdot w_0 = 0$$

When the deformations are small, the neo-Hookean material obeys Hook's law with the shear modulus $\mu = 2c_1$ and Poisson's ratio $\nu = 0.5$

$$\varepsilon = {}^1/_2\mu^{-1}\,[\mathrm{T} - {}^1/_3 I_1(\mathrm{T})\,\mathrm{E}]$$

when the faces of the plate are stress-free

$$\varepsilon = {}^1/_2\mu^{-1}\,(\mathrm{T} - {}^1/_3 T\cdot\cdot\mathrm{E}_2\mathrm{E})$$

If in addition the plate is very thin, the quantity ${}^{11}/_5 h^2\nabla^2$ will be small compared with 12 and can be neglected. The equation for the bending w_0 will then assume the form

$${}^1/_3 h^3\mu\nabla^4 w_0 - \nabla\cdot\mathrm{T}\cdot\nabla w_0 = 0$$

When $\nu = 0.5$ the cylindrical rigidity of the plate is

$$D\frac{Eh^3}{12(1-\nu^2)} = \frac{2\mu(1+\nu)}{12(1-\nu^2)}h^3 = \frac{h^3}{3}\mu$$

Thus for small initial deformation the basic differential equation of buckling (3.12) transforms into the equation of the classical theory of stability of plates [4].

4. Axially symmetric buckling of a uniformly compressed circular plate. (Example 1). Let r, θ denote the polar coordinates in the plane of the undeformed plate and e_i, e_θ the corresponding basis vectors. With the axial symmetry present, the solution of (3.11) is

$$w_0 = CJ_0(\alpha r) + D, \qquad v = \mathbf{w}_2\cdot e_\theta = 0, \qquad \alpha^2 = -\frac{60(1-G_1^{-3})}{h^2(1+3G_1^{-3})(6-G_1^{-3})}$$

$$u = \mathbf{w}_2\cdot\mathbf{e}_r = {}^1/_5\,G_1(6-G_1^{-3})\,C\alpha J_1(\alpha r)$$

Here J_0 and J_1 are Bessel functions, and C and D are constants of integration. If the plate is clamped along the contour, then the condition $u = 0$ when $r = a$ leads to the equation

$$\frac{60(G_1^3 - G_1^6)}{6G_1^6 + 17G_1^3 - 3} = \Upsilon_n^2\frac{h^2}{a^2} \tag{4.1}$$

where γ_n are the zeros of J_1 ($n = 1, 2, \ldots$). It can easily be confirmed that when (4.1) holds, so does

$$\mathbf{e}_r\cdot(\mathrm{V}_1 + \mathrm{V}_3) = 0 \text{ when } r = a$$

i.e. this solution holds when the clamped edge is free to slide.

To compare the results we shall consider the exact solution of the axially symmetric bifurcation of equilibrium of a circular cylinder made of neo-Hookean material, uniformly compressed over its side surface. The faces of the cylinder are free and the side surface, although fixed with respect to rotation, may slide in the direction of the z-axis (axis of the cylinder). When the cylinder is thin, the above boundary conditions become applicable to a clamped plate. We require therefore solutions of (1.8) and (1.9) satisfying the conditions

$$\mathbf{i}_3\cdot\mathrm{D}^\bullet = 0 \quad (z = \pm{}^1/_2 h), \qquad \mathbf{e}_r\cdot D^\bullet\cdot\mathbf{i}_3 = 0, \quad u = \mathbf{w}\cdot\mathbf{e}_r = 0 \quad (r=a) \tag{4.2}$$

Here $\mathrm{D}^\bullet$ is given by (1.7). In the present case of uniform compression we have

$$\mathrm{G}^\times = G_1\mathrm{E}_2 + G_3\mathbf{i}_3\mathbf{i}_3 = \beta^2\mathrm{E}_2 + \beta^{-4}\mathbf{i}_3\mathbf{i}_3$$

From (1.8) and (1.9) follows

$$\nabla\cdot\mathrm{G}^{\times-1}\cdot\nabla p = 0 \tag{4.3}$$

where ∇ is the three-dimensional Hamiltonian.

For axisymmetric forms of bifurcation we have $v = w\cdot e_\theta = 0$ and Eqs. (1.8) and (1.9) become

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{\beta}\frac{\partial p}{\partial r} = 0 \tag{4.4}$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} + \beta^2\frac{\partial p}{\partial z} = 0, \qquad \frac{\partial u}{\partial r} + \frac{u}{r} + \beta^3\frac{\partial w}{\partial z} = 0 \qquad (w = \mathbf{w}\cdot\mathbf{i}_3)$$

Equation (4.3) can be written as

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\beta^6\frac{\partial^2 p}{\partial z^2}=0 \tag{4.5}$$

Using (1.7) and taking into account

$$m=G_3=\beta^{-4}$$

we find (4.2) replaced by

$$u=0,\qquad \frac{\partial u}{\partial z}+\beta^3\frac{\partial w}{\partial r}=0 \qquad (r=a) \tag{4.6}$$

$$\frac{\partial u}{\partial z}+\frac{1}{\beta^3}\frac{\partial w}{\partial r}=0,\qquad 2\frac{\partial w}{\partial z}+p\beta^2=0 \qquad (z=\pm{}^1/_2 h) \tag{4.7}$$

We shall seek only those bending forms of bifurcation, for which u and p are odd functions of z, and w is an even function of z. Then

$$p=BJ_0(kr)\,\mathrm{sh}\,k\beta^{-3}r \qquad (B,\ k=\mathrm{const})$$

is a solution of (4.5).

From the first two equations of (4.4) we now obtain

$$w=\left[\frac{\beta^5}{(1-\beta^6)k}B\,\mathrm{ch}\,k\beta^{-3}z+A\,\mathrm{ch}\,kz\right]J_0(kr)+D$$

$$u=\left[\frac{\beta^5}{(1-\beta^6)k}B\,\mathrm{sh}\,k\beta^{-3}z+A'\,\mathrm{sh}\,kz\right]J_1(kr)$$

and from the third equation of (4.4) we have

$$A'=-\beta^{-3}A$$

The boundary conditions (4.6) will hold on the side surface if the constant k is chosen such that $J_1(ka)=0$. Thus let us set

$$k=k_n=\gamma_n/a \qquad (n=1,2,3,\ldots)$$

Consequently we have the following solution for every n:

$$u_n=\left[\frac{\beta^5}{(1-\beta^6)k_n}B_n\,\mathrm{sh}\,k_n\beta^{-3}z-A_n\beta^3\,\mathrm{sh}\,k_nz\right]J_1(k_nr)$$

$$w_n=\left[\frac{\beta^5}{(1-\beta^6)k_n}B_n\,\mathrm{ch}\,k_n\beta^{-3}z+A_n\,\mathrm{ch}\,k_nz\right]J_0(k_nr)+D$$

$$p_n=B_n\,\mathrm{sh}\,k_n\beta^{-3}z\,J_0(k_nr)$$

Further, assuming that conditions (4.7) hold on the cylinder endfaces, we construct a system of homogeneous linear equations in the constants A_n and B_n and equate its determinant to zero, thus arriving at the following transcendental equations for the critical values of β

$$\frac{(1+\beta^6)^2}{\beta^3}\,\mathrm{th}\left(\frac{1}{2}\gamma_n\frac{h}{a}\beta^{-3}\right)=4\,\mathrm{th}\left(\frac{1}{2}\gamma_n\frac{h}{a}\right) \tag{4.8}$$

When $\gamma_n h/a=1$, the root of (4.8) is $\beta_*=0.944$. Since $\gamma_1=3.83$, the above value corresponds to the first form of plate buckling where the ratio $2a/h=7.66$. Equation (4.1) of the approximate two-dimensional theory yields for $\gamma_n h/a=1$, $\beta_*=0.945$. The classical theory of buckling of plates in this case gives

$$\beta_*=1-\frac{1}{18}\gamma_n^2\frac{h^2}{a^2} \qquad (\beta_*=0.944 \quad \text{when} \quad \gamma_n h/a=1)$$

Thus, if the uniformly compressed plate is sufficiently thin, both, the two-dimensional theory nonlinear with respect to the initial deformation and the classical theory of buckling, give correct results for the critical deformation corresponding to the first form of buckling.

We note that according to the exact solution of (4.8) the critical value of the initial compression corresponding to an infinite number of nodal contours ($\gamma_n \to \infty$) is given by

$$(1 + \beta^6)^2 = 4\beta^3$$

from which we obtain $\beta_\infty = 0.664$. This also implies that a plate, no matter how thick, will buckle when $\beta = \beta_\infty$.

5. Rectangular plate under a combined initial load (Example 2). Let the initial deformation of the plate be such, that its principal axes e_s are parallel to the sides of the plate

$$G^\times = \lambda_1^2 i_1 i_1 + \lambda_2^2 i_2 i_2 + \lambda_1^{-2}\lambda_2^{-2} i_3 i_3$$

Then the system of differential equations (3.11) can be written as

$$\frac{h^2}{10}\left(\lambda_1^2 + \frac{3}{\lambda_1^2\lambda_2^2}\right)\frac{\partial^2 u}{\partial x^2} + \frac{h^2}{10}\lambda_1^2\frac{\partial^2 u}{\partial y^2} + \frac{h^2}{10}\frac{3}{\lambda_1^2\lambda_2^2}\frac{\partial^2 v}{\partial x \partial y} - \lambda_1^2 u - \frac{1}{\lambda_1\lambda_2}\frac{\partial w_0}{\partial x} = 0$$

$$\frac{h^2}{10}\left(\lambda_2^2 + \frac{3}{\lambda_1^2\lambda_2^2}\right)\frac{\partial^2 v}{\partial y^2} + \frac{h^2}{10}\lambda_2^2\frac{\partial^2 v}{\partial x^2} + \frac{h^2}{10}\frac{3}{\lambda_1^2\lambda_2^2}\frac{\partial^2 u}{\partial x \partial y} - \lambda_2^2 v - \frac{1}{\lambda_1\lambda_2}\frac{\partial w_0}{\partial y} = 0$$

$$\frac{5}{\lambda_1^2\lambda_2^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(6 - \frac{1}{\lambda_1^4\lambda_2^2}\right)\frac{\partial^2 w_0}{\partial x^2} + \left(6 - \frac{1}{\lambda_1^2\lambda_2^4}\right)\frac{\partial^2 w_0}{\partial y^2} = 0 \qquad (5.1)$$

Here x and y are Cartesian coordinates in the neutral plane of the undeformed plate, $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$, while u and v denote the x- and y-components of the vector w_2. We assume that the plate is hinged along all its edges, i.e. that it has the following boundary conditions

$$w_0 = 0, \quad M_{11} = 0, \quad v = 0 \quad \text{when } x = 0, \quad x = a$$
$$w_0 = 0, \quad M_{22} = 0, \quad u = 0 \quad \text{when } y = 0, \quad y = b$$

Using (3.7) we can easily confirm that the corresponding functions are

$$u = U\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y, \qquad v = V\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y$$

$$w_0 = W\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y$$

Here U, V, W are independent of x and y, and $m, n = 1, 2, 3, \ldots$ Inserting these expressions into (5.1) we arrive at a system of homogeneous linear equations in U, V and W. Calculating its determinant for a square ($a = b$) plate we obtain

$$[h_*^2(m^2+n^2)+1]\left\{-6\left(\frac{m^2}{\lambda_1^2}+\frac{n^2}{\lambda_2^2}\right) + 6\lambda_1^2\lambda_2^2(m^2+n^2)[h_*^2(m^2+n^2)+1] + \right.$$
$$\left. + 17h_*^2(m^2+n^2)\left(\frac{m^2}{\lambda_1^2}+\frac{n^2}{\lambda_2^2}\right) - 3\frac{h_*^2}{\lambda_1^2\lambda_2^2}\left(\frac{m^2}{\lambda_1^2}+\frac{n^2}{\lambda_2^2}\right)^2\right\} = 0 \qquad (5.2)$$

$$\left(h_*^2 = \frac{\pi^2}{10}\frac{h^2}{a^2}\right)$$

If we further set

$$\left(\frac{m^2}{\lambda_1^2}+\frac{n^2}{\lambda_2^2}\right)\lambda_1^{-2}\lambda_2^{-2} = t \qquad (5.3)$$

then we can reduce (5.2) to a quadratic equation which, in turn, yields

$$t = \frac{1}{6h_*^2}\{-[6-17h_*^2(m^2+n^2)] \pm$$
$$\pm \sqrt{[6-17h_*^2(m^2+n^2)]^2 + 72h_*^2(m^2+n^2)[1+h_*^2(m^2+n^2)^2]} \qquad (5.4)$$

Since t cannot be negative, we must disregard the root with the minus sign before the radical. The other root is always positive, therefore a critical value of t exists for any thickness of the plate. Having found this value for given m and n, we can use (5.3) to obtain the critical relation between the parameters λ_1 and λ_2.

The pressures p_1, p_2 acting on the boundaries $x = 0$, $x = a$ and $y = 0$, $y = b$ in the initial deformed state and calculated per unit area of the undeformed body are expressed, according to (1.3), by

$$p_1^* = \frac{p_1}{4c_1} = \frac{1}{2\lambda_1}\left(\frac{1}{\lambda_1^2\lambda_2^2} - \lambda_1^2\right), \qquad p_2^* = \frac{p_2}{4c_1} = \frac{1}{2\lambda_2}\left(\frac{1}{\lambda_1^2\lambda_2^2} - \lambda_2^2\right)$$

The critical relation between λ_1 and λ_2 defines a curve in the plane of load parameters p_1^* and p_2^*, for every set of integers m and n.

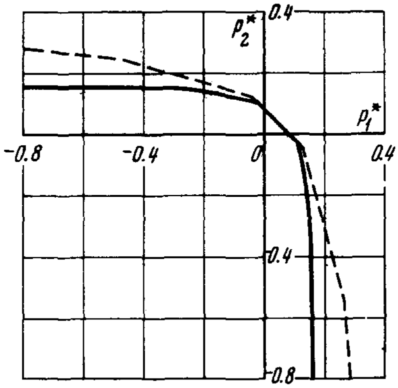

Fig. 1

The line composed of segments of these curves nearest to the coordinate origin $p_1^* = 0$, $p_2^* = 0$, separates the regions of stability and instability, and is therefore called the boundary line.

Such a line computed according to the formulas (5.3) and (5.4) is shown in Fig. 1. The broken line corresponds to the classical theory of buckling of plates. In our calculations we have used the value of $h_*^2 = 0.01$ which corresponds approximately to the plate thickness of $h/a = 0.1$. It follows therefore that when the plate is compressed in both directions, then for the lower critical loads the conventional theory of buckling yields nearly correct results; if, on the other hand, the plate is stretched in one direction and compressed in the other, the conventional theory gives excessive results.

Authors of [5, 6] have deduced that, when a rectangular plate made of neo-Hookean material or of Mooney material is uniformly compressed, then a limiting thickness exists above which the plate remains stable under any amount of compression. However all known exact solutions of the problems on bifurcation of equilibrium of solid elastic bodies point to the opposite: if the initial deformation reaches a certain finite value, the plate will lose its stability irrespective of its thickness.

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